

Electromagnetic induction: physics and flashbacks

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Abstract. A general law for electromagnetic induction phenomena is derived from Lorentz force and Maxwell equation connecting electric field and time variation of magnetic field. The derivation provides with a unified mathematical treatment the statement according to which electromagnetic induction is the product of two independent phenomena: time variation of magnetic field and effects of magnetic field on moving charges. The “flux rule” has a restricted validity; furthermore, it is neither a causal law nor a “good” field-law. A survey of Faraday and Maxwell writings shows that the taking root of the “flux rule” constitutes an open historical problem.

Riassunto. Partendo dalla forza di Lorentz e dall’equazione di Maxwell che lega il campo elettrico alla variazione temporale del campo magnetico si ottiene una legge generale dell’induzione elettromagnetica. Questa legge dà forma matematica all’affermazione secondo cui l’induzione elettromagnetica è il prodotto di due fenomeni indipendenti: la variazione temporale del campo magnetico e gli effetti del campo magnetico su cariche in moto. La “legge del flusso” ha validità limitata; inoltre, essa non è né una legge causale né una “buona” legge di campo. Un’analisi dei lavori di Faraday e Maxwell mostra come il radicamento della “legge del flusso” costituisca un problema storico aperto.

1. Introduction

It is, in general, acknowledged that the theoretical treatment of electromagnetic induction phenomena presents serious problems when part of the induced circuit is moving. Some authors speak of exceptions to the “flux rule”¹, others save the “flux rule” by *ad hoc* choices of the integration line over which the induced *emf* is calculated. Several attempts to overcome these difficulties have been made; a comprehensive one has been performed by Scanlon, Henriksen and Allen.² However, their treatment uses an *ad hoc* modification of the usual definition of induced *emf* and - as others - fails to recognize that one must distinguish between the velocity of the circuit elements and the velocity of the electrical charges contained in them (see section 3 below). Therefore, these authors reestablish the “flux rule” and, consequently, do not solve the problems posed by its application.

Since 1992, I have been teaching electromagnetism in a course for Mathematics students and I had to deal with the problems outlined above. I have found that it is possible to get a general law for electromagnetic induction that contains the standard “flux rule” as a particular case.

The matter has conceptual relevance; it has also historical and epistemological aspects that deserve to be investigated. Therefore, it is, perhaps, worthwhile to submit the following considerations to the attention of a public wider than that of my students.

Textbooks show a great variety of positions about how the “flux rule” can be applied to the known experimental phenomena of electromagnetic induction. Among the more lucid approaches, let us refer to the treatment given by Feynman, Leighton and Sands in the *Feynman Lectures on Physics*. They write:

¹ R. Feynman, R. Leighton and M. Sands, *The Feynman Lectures on Physics*, vol. II, (Addison Wesley, Reading, Ma., 1964), pp. 17-2,3.

² J. Scanlon, R.N. Henriksen and J.R. Allen, “Approaches to electromagnetic induction”, *Am. J. Phys.*, 37, (1969), 698-708.

In general, the force per unit charge is $\vec{F} / q = \vec{E} + \vec{v} \times \vec{B}$. In moving wires there is the force from the second term. Also, there is an \vec{E} field if there is somewhere a changing magnetic field. They are two independent effects, but the *emf* around the loop of wire is always equal to the rate of change of magnetic flux through it.³

This sentence is followed by a paragraph entitled *Exceptions to the “flux rule”*, where the authors treat two cases - the Faraday disc and the “rocking plates” – both characterized by the fact that, in a part of the circuit, the *material of the circuit* is changing. As the authors put it, at the end of the discussion:⁴

The “flux rule” does not work in this case. It must be applied to circuits in which the *material* of the circuit remain the same. When the material of the circuit is changing, we must return to the basic laws. The *correct* physics is always given by the two basic laws

$$\begin{aligned}\vec{F} &= e(\vec{E} + \vec{v} \times \vec{B}) \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}\end{aligned}$$

The statement “The *correct* physics is always given by the two basic laws”, if coherently developed, leads to a general law of electromagnetic induction: as a matter of fact, this has been the starting idea for the treatment given below.

2. Maxwell and Lorentz

Let us begin by stressing that the expression of Lorentz force

$$\vec{F} = q(\vec{E} + \vec{v}_{charge} \times \vec{B}) \quad (1)$$

not only gives meaning to the fields solutions of Maxwell equations when applied to point charges, but yields new predictions.⁵

The velocity appearing in the expression of Lorentz force is the velocity of the charge: from now on, we shall use the symbol \vec{v}_{charge} for distinguishing the charge velocity from the velocity \vec{v}_{line} of the circuit element that contains the charge. This is a basic point of the present treatment.

3. A definition of induced *emf*

The standard definition of the *emf* that can be found in textbooks is

$$emf = \oint_l \vec{E} \cdot d\vec{l} \quad (2)$$

This assumption works well when the induced *emf* - in a circuit at rest with respect to the inertial frame of the laboratory - is due to a time variation of the magnetic field . But if the *emf* is *induced*

³ Note 1, p. 17-2.

⁴ Ibidem, p. 17-3.

⁵ The fact that the expression of Lorentz force can be derived by considering an inertial frame in which the charge is at rest and by *assuming* that the force acting on it is simply given by $\vec{F} = q\vec{E}$, does not change the matter.

by any of the induction devices used for producing electrical energy, we all know that the electrical energy in the induced circuit comes from the mechanical energy necessary for making the induction device work. In these cases, the magnetic field plays an *indispensable* and *intermediary* role: it allows the mechanical energy to be converted into electrical energy. Therefore, it is not wise to leave it out from our theoretical description.

Of course, these considerations serve only as a sound support for a new definition of the induced *emf*

$$emf = \oint_l (\vec{E} + \vec{v}_{charge} \times \vec{B}) \cdot d\vec{l} = \oint_l \vec{E} \cdot d\vec{l} + \oint_l (\vec{v}_{charge} \times \vec{B}) \cdot d\vec{l} \quad (3)$$

i.e. as the line integral around the circuit of the Lorentz force acting on a unit positive charge. The integral of equation (3) is the work done by the electromagnetic field on a unit positive charge along the closed path considered: therefore, it appears as the *natural* definition of *emf*.

Now, according to Maxwell equations:

$$\vec{E} = -grad\varphi - \frac{\partial \vec{A}}{\partial t} \quad (4)$$

where φ and \vec{A} are the scalar and vector potential. Then, equation (3) becomes:

$$emf = -\oint_l \frac{\partial \vec{A}}{\partial t} \cdot d\vec{l} + \oint_l (\vec{v}_{charge} \times \vec{B}) \cdot d\vec{l} \quad (5)$$

since the integral of $grad\varphi$ around a close loop is zero. This is the *general law* for electromagnetic induction.

It says that:

1. The induced *emf* is, in general, given by the sum of two terms.
2. The first term is due to the time variation of the vector potential and, therefore, to the time variation of the magnetic field. This term vanishes when \vec{A} and \vec{B} are constant.
3. The second term is due to the magnetic component of Lorentz force.
4. Therefore, the electromagnetic induction has *two distinct causes*: time variation of the magnetic field *and* motion of charges in magnetic fields.

Since the charge velocity is given by⁶

$$\vec{v}_{charge} = \vec{v}_{line} + \vec{v}_{drift}$$

the complete expression of the general law is

$$emf = -\oint_l \frac{\partial \vec{A}}{\partial t} \cdot d\vec{l} + \oint_l (\vec{v}_{line} \times \vec{B}) \cdot d\vec{l} + \oint_l (\vec{v}_{drift} \times \vec{B}) \cdot d\vec{l} \quad (6)$$

⁶ Since both \vec{v}_{line} and \vec{v}_{drift} are small with respect to the light velocity, we can use the galileian composition rule for the velocities.

Of course, the general law can be written in terms of the magnetic field. By applying Stokes theorem to the first integral of equation (6) and by using the relation

$$\oint_S \frac{\partial \vec{B}}{\partial t} \cdot \hat{n} dS = \frac{d}{dt} \oint_S \vec{B} \cdot \hat{n} dS + \oint_l (\vec{v}_{line} \times \vec{B}) \cdot d\vec{l} \quad (7)$$

that holds for every vector field whose divergence vanishes⁷ - we get

$$emf = \left[-\frac{d}{dt} \int_S \vec{B} \cdot \hat{n} dS - \oint_l (\vec{v}_{line} \times \vec{B}) \cdot d\vec{l} \right] + \oint_l (\vec{v}_{line} \times \vec{B}) \cdot d\vec{l} + \oint_l (\vec{v}_{drift} \times \vec{B}) \cdot d\vec{l} \quad (8)$$

We have grouped the first two integrals of equation (8) under square brackets for underlying their common mathematical and physical origin: their sum *must* be zero when the magnetic field does not depend on time. Equation (8) shows that:

- A. The “flux rule” has restricted validity. It is valid only when the induced circuit is at rest with respect to the inertial frame of the laboratory *and* the contribution of the drift velocity vanishes.
- B. When the magnetic field is constant and the integral containing the drift velocity vanishes, the “flux rule” predicts the correct value of the induced *emf* (because the two integrals containing the line velocity cancel each other): however, *physically*, it is the sum of the first two integrals that must be zero and not the sum of the two line integrals containing the velocity of the line elements.

The general law (6) – in the case of a circuit made by a loop of wire – assumes the form

$$emf = -\oint_l \frac{\partial \vec{A}}{\partial t} \cdot d\vec{l} + \oint_l (\vec{v}_{line} \times \vec{B}) \cdot d\vec{l} \quad (9)$$

because, in a loop of wire, the drift velocity is parallel to $d\vec{l}$. However, *in extended materials, the contribution of the drift velocity cannot be ignored* (see below the discussion of Corbino and Faraday disc).

4. Good fields, bad fields and causality

The general law (9) expressed in terms of the vector potential, is a causal law: the induced electric field $\vec{E}_i = -\partial \vec{A} / \partial t + \vec{v}_{charge} \times \vec{B}$, at the generic instant t , “produces” the induced *emf* at the same instant t . The vector potential is, in this case, a “good” field, because it enters the local equation that yields the induced electric field.

On the contrary, *the “flux rule” is not a causal law*. In fact, it relates the induced *emf* to a time variation of the magnetic field on an *arbitrary* surface that has the circuit as a contour. Moreover,

⁷ See, for instance: A. Sommerfeld, *Lectures in theoretical Physics*, Academic Press, (1950), vol. II, pp. 130-132; ibidem, vol. III, p. 286.

the \vec{B} field should “act at a distance” with infinite velocity on the points of the circuits for “producing” the induced emf . In this situation, the magnetic field is a “bad” field.⁸

5. Is the concept of emf necessary?

Yes, it is. In spite of the fact that, at a first glance, we could dispense with it by using the concept of induced electric field. To convince ourselves, let us consider the simple case in which only a part of the induced circuit is moving in a uniform and constant magnetic field: the electric field is induced only in the moving part; however, the induced electric current depends also on the physical properties of those parts of the circuit that are at rest. As a matter of fact the induced current can be predicted only by writing that $I = emf / R$, where R is the *total* resistance of the circuit.

This remind us of another widely discussed problem, i.e. the “localization” of the induced emf . The question “where is localized the induced emf ?” is a meaningful one. The application of the general law automatically answers this question (while, of course, the “flux rule” does not): being a local law, the general law says where the electric field is induced and where it is not. In section 7.a we shall illustrate this point.

6. How the general law works

In a previous paper, several particular cases have been discussed: here, we shall comment only on some of them by referring, for the details, to the treatment given there.⁹

- a. **A conducting bar**, of length a , slides at constant velocity over an U shaped conducting frame in a uniform and constant magnetic field (Fig. 1).

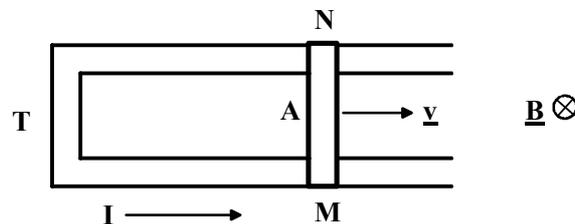


Fig. 1: the moving bar.

The magnetic field is perpendicular to and entering the page. The emf is induced in the moving bar, and its value is given by vBa where v is the velocity of the bar and B the magnetic field. In this case, the localization of the emf has the following operational meaning: in the bar the current enters from the point at lower potential, while in any portion of the U shaped conductor, the current enters from the point at higher potential. The discussion explains why, in this case, the “flux rule” yields the correct predictions (see point B in section 3 above) but, of course, it is not able to say where the induced emf is localized.¹⁰ If we consider the circuit of the figure at a certain instant, we have an example of a dc current circuit in which the moving bar acts as a battery: the theoretical treatment of such a circuit is completely transparent, because we can easily manage the electric fields operating into the bar. It is a good application of the general law for teaching purposes.¹¹

⁸ A similar discussion about “real” or “not real” fields can be found in: R. Feynman, R. Leighton and M. Sands, *The Feynman Lectures on Physics*, vol. II, (Addison Wesley, Reading, Ma., 1964), vol. II, pp. 15-7, 15-8.

⁹ G. Giuliani, *On electromagnetic induction*, http://matsci.unipv.it/percorsi/emi_web.pdf; or: <http://babbage.sissa.it/abs/physics/0008006>.

¹⁰ Ibidem, pp. 9-10.

¹¹ G. Giuliani, I. Bonizzoni, *Lineamenti di elettromagnetismo*, to be published.

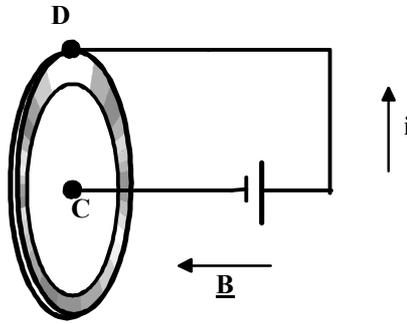


Fig. 2: Corbino disc.

- b. **Corbino's disc.** In 1911, Corbino studied theoretically and experimentally the case of a conducting disc with a hole at its center (Fig. 2). If a voltage is applied between the inner and the outer periphery of the disc, a radial current will flow, provided that the experimental setup is realized in a way suitable for maintaining the circular symmetry: the inner and outer periphery are covered by highly conducting electrodes; therefore, the inner and outer periphery are equipotential lines. Then, if a uniform and constant magnetic field is applied perpendicularly to the disc, a circular current will flow in the disc.¹²

The discussion shows how the charge drift velocity plays its role in the building up of the induced electromotive field and, therefore, of the induced *emf*. The application of the general law to the Corbino disc is interesting for the following reasons:¹³

- No part of the circuit is moving and the magnetic field is uniform and *constant*. Therefore, according to the “flux rule”, there is no induced *emf*.
 - According to the general law, there is an induced circular *emf*, due to the radial drift velocity of the charges. This induced circular *emf* produces a circular current. The circular induced *emf* is “distributed” homogeneously along a circle. Each circular strip of section sdr (s is the thickness of the disc and dr an infinitesimal portion of the radius) acts as a battery that produces a current in its own resistance: therefore, the potential difference between two points arbitrarily chosen on a circle is zero. Hence, as it must be, each circle is an equipotential line. The observed phenomenon may be described as due to an increased radial resistance: this is the magnetoresistance effect. The treatment of Corbino disc by the general law yields magnetoresistance effects without using a microscopic approach based on scattering processes. This discussion shows how the general law can be applied to phenomena traditionally considered outside the phenomenological domain for which it has been derived.
- c. **Faraday's disc.** A conducting disc rotates with constant angular velocity ω in a uniform and constant magnetic field B perpendicular to the plane of the disc. A conducting frame makes conducting contacts with the center and a point on the periphery of the disc.

¹² O.M. Corbino, “Azioni elettromagnetiche dovute agli ioni dei metalli deviati dalla traiettoria normale per effetto di un campo”, *Il Nuovo Cimento* 1, 397-419 (1911). A german translation of this paper appeared in *Phys. Zeits.*, 12, 561-568 (1911). For a historical reconstruction see: S. Galdabini and G. Giuliani, “Magnetic field effects and dualistic theory of metallic conduction in Italy (1911-1926): cultural heritage, epistemological beliefs, and national scientific community”, *Ann. Science* 48, 21-37 (1991). As pointed out by von Klitzing, the quantum Hall effect may be considered as an ideal (and quantized) version of the Corbino effect corresponding to the case in which the current in the disc, with an applied radial voltage, is only circular: K. von Klitzing, “The ideal Corbino effect” in: P.E. Giua ed., *Commemorazione di Orso Mario Corbino*, Centro Stampa De Vittoria, Roma, 1987), pp. 43-58.

¹³ Ref. 9, pp. 16-19.

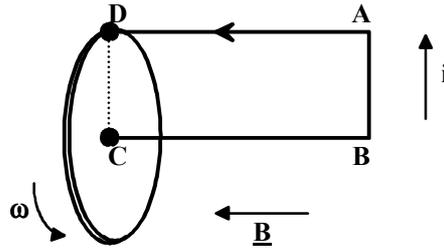


Fig. 3: Faraday disc.

A preliminary discussion - in which the drift velocity is neglected - shows how the general law and “flux rule” deal with this case. The “flux rule” predicts the correct result only by assuming *ad hoc* integration lines.¹⁴ As a second step, using the results obtained with the Corbino disc, the Faraday disc - with circular symmetry - is treated taking into account also the drift velocity. Faraday has found that there is an induced current also if the disc and the cylindrical magnet that produces the field are rotated together, i.e. in a situation in which there is no relative motion between the conductor and the magnet (see below section 8.1). This result is easily explained by the general law (and, using *ad hoc* hypothesis, also by the “flux rule”). As we shall see, Faraday explained this and other similar results by stating that there is an induced current only if there is relative motion between the conductor and the lines of magnetic force and that the lines do not rotate when the magnet does. In the inertial laboratory frame, the axis of the magnet is at rest when the magnet rotates: in the same reference frame, also Faraday’s lines of magnetic force are at rest.

In our previous paper, one can find also a discussion of the unipolar induction, of a case proposed by Kaempffer¹⁵ and of the “rocking plates” discussed in Feynman Lectures.¹⁶

7. Possible new experimental tests

The fact the general law of electromagnetic induction correctly describes the physics of Corbino’s disc, must be considered as an experimental corroboration of the law.

In a discussion about possible new experimental tests, Attlio Rigamonti (a colleague of mine Department) has suggested the following experiment.

Take a copper ring and cover it with a sufficiently thick layer of a superconductor: an applied magnetic field will non enter the copper, owing to the shielding action of the superconductor. Now, if we turn a magnetic field on, according to the general law, which is a local law, there will be no induced current in the copper ring, because the magnetic field and the vector potential cannot enter the copper: in the copper ring, $\partial \vec{B} / \partial t = \partial \vec{A} / \partial t = 0$. The “flux rule” instead predicts an induced current because the magnetic flux through the copper ring changes when we turn the magnetic field on.

8. Why the “flux rule”?

The establishment of the general law for the electromagnetic induction poses a puzzling question: how things have gone from a historical point of view?

¹⁴ Ibidem, pp. 11-12.

¹⁵ F. A. Kaempffer, *Elements of Physics*, Blaisdell Publ. Co., Waltham, Mass., 1967, p. 164; quoted by Scanlon, Henriksen and Allen.

¹⁶ Ref. 1, pp. 17-3.

We shall not engage in a historical reconstruction; more simply, we are looking for hints and suggestions that, if taken into account, would have thrown some doubts about the general validity of the “flux rule”.

8.1 Faraday

Starting 1851, Faraday, began to systematically develop an idea already conceived twenty years before, during his first researches on electromagnetic induction: the lines of magnetic force. In his words:

From my earliest experiments on the relation of electricity and magnetism (114, note), I have had to think and speak of lines of magnetic force as representations of the magnetic power; not merely in the points of quality and direction, but also in quantity. The necessity I was under of a more frequent use of the term in some recent researches (2149, &c), has led me to believe that the time has arrived, when the idea conveyed by the phrase should be stated very clearly, and should also be carefully examined [...].¹⁷

Faraday distinguishes between two conceptions of lines of force:

I have recently been engaged in describing and defining the lines of magnetic force (3070), i.e. those lines which are indicated in a general manner by the disposition of iron filings or small magnetic needles, around or between magnets; and I have shown, I hope satisfactorily, how these lines may be taken as exact representants of the magnetic power, both as to disposition and amount; also how they may be recognized by a moving wire in a manner altogether different in principle from the indications given by a magnetic needle, and in numerous cases with great and peculiar advantages. The definition then given had no reference to the physical nature of the force at the place of action, and will apply with equal accuracy whatever that may be; and this being very thoroughly understood, I am now about to leave the strict lines of reasoning for a time, and enter upon a few speculations respecting the physical character of the lines of force, and the manner in which they may be supposed to be continued through space. We are obliged to enter into such speculations with regard to numerous natural powers, and, indeed, that of gravity is the only instance where they are apparently shut out.¹⁸

The lines of forces may be either only a “theoretical entity” used for describing the phenomena or they may have also physical existence, i.e., they may be “real entities” in the world:

I have not referred in the foregoing considerations to the view I have recently supported by experimental evidence, that the lines of force, *considered simply as representants of the magnetic power* (3117), are closed curves, passing in one part of their course through the magnet, and in the other part through the space around it. These lines are identical in their nature, qualities and amount, both within the magnet and without. *If to these lines*, as formerly defined (3071), *we add the idea of physical existence*, and then reconsider such of the cases which have just been mentioned as come under the new idea, it will be seen at once that the probability of curved external lines of force, and therefore of the physical existence of the lines, is as great, and even far greater than before. [...]¹⁹

And:

¹⁷ M. Faraday, *Experimental Researches in Electricity*, vol. III, London, 1855, p. 328, (3070). In all quotations of Faraday *Researches*, and - later - of Maxwell *Treatise*, we put under brackets the number of the paragraph.

¹⁸ Ibidem, p. 407- 408, (3243)

¹⁹ Ibidem, p. 417, (3264). My italics.

Having applied the term *lines of magnetic force* to an abstract idea, which I believe represents accurately the nature, condition, direction, and comparative amount of the magnetic forces, without reference to any physical condition of the force, I have now applied the term *physical lines of force* to include the further idea of their physical nature. The first set of lines I *affirm* upon evidence of strict experiment (3071, &c). The second set of lines I advocate, chiefly with a view of stating the question of their existence; and though I should not have raised the argument unless I had thought it both important, and likely to be answered ultimately in the affirmative, I still hold the opinion with some hesitation, with as much, indeed, as accompanies any conclusion I endeavour to draw respecting points in the very depths of science, as, for instance, regarding one, two, or no electric fluids; or the real nature of a ray of light, or the nature of attraction, even that of gravity itself, or the general nature of matter.²⁰

Faraday clearly states that the conformity of a theory with experiment can be ascertained; on the other hand, the assertion of the real existence of a theoretical entity used by a theory – that is in conformity with experiment – cannot be held as certain.

According to Faraday, *there is an induced current in a conducting wire when there is relative motion between the conducting wire and the lines of magnetic force*. This is true not only when a conductor moves towards (away from) a magnet or a magnet moves towards (away from) a conductor; but also when a current in a circuit, that is at rest with respect the conductor, varies with time:

In the first experiments (10, 13), the inducing wire and that under induction were arranged at a fixed distance from each other, and then an electric current sent through the former. In such cases the magnetic curves themselves must be considered as moving (if I may use the expression) across the wire under induction, from the moment at which they begin to be developed until the magnetic force of the current is at its utmost; expanding as it were from the wire outwards, and consequently being in the same relation to the fixed wire under induction, as if it had moved in the opposite direction across them, or towards the wire carrying the current. Hence the first current induced in such cases was in the contrary direction to the principal current (17, 235). On breaking the battery contact, the magnetic curves (which are mere expressions for arranged magnetic forces) may be conceived as contracting upon and returning towards the failing electrical current, and therefore move in the opposite direction across the wire, and cause an opposite induced current in the first.²¹

And:

When *lines of force* are spoken of as crossing a conducting circuit (3087), it must be considered as effected by the *translation* of a magnet. No mere rotation of a bar magnet on its axis, produces any induction effect on circuits exterior to it; for then, the conditions above described (3088) are not fulfilled. The system of power about the magnet must not be considered as necessarily revolving with the magnet, any more than the rays of light which emanate from the sun are supposed to revolve with the sun. The magnet may even, in certain cases (3097), be considered as revolving amongst its own forces, and producing a full electric effect, sensible at the galvanometer.²²

As a matter of fact:

To prove the point with an ordinary magnet, a copper disc was cemented upon the end of a cylinder magnet, with paper intervening; the magnet and disc were rotated together, and collectors (attached to the galvanometer) brought in contact with the circumference and the central part of the copper plate. The galvanometer needle moved as in the former cases, and the *direction* of motion was the *same* as that which would have resulted, if copper only had revolved, and the

²⁰ Ibidem, p. 437, (3299).

²¹ M. Faraday, *Experimental Researches in Electricity*, vol. I, London, 1839, (238).

²² M. Faraday, *Experimental Researches in Electricity*, vol. III, London, p. 336-337, (3090).

magnet been fixed. Neither was there any apparent difference in the quantity of deflection. Hence, rotating the magnet, causes no difference in the results; for a rotatory and stationary magnet produce the same effect upon the moving copper.^{23, 24}

This result is of paramount relevance: not only it is the best experimental support of the statement n. 3090 reported just above; it is also a strong indication in favor of the physical existence of Faraday's lines of magnetic force.²⁵

Faraday's theoretical description of electromagnetic induction phenomena is characterized by the following features:

- it is a field theory, in the sense that the physical interactions are local interactions between conductors and lines of magnetic force
- the cause of induction phenomena is unique: the intersection of elements of conductors with lines of magnetic force
- its basic law is: there is an induced current in a conducting wire *only* when there is *relative motion between the conducting wire and the lines of magnetic force*.

An interesting question is: how could this basic law be translated into a formula? Usually, it is held that the "flux rule" is the mathematical translation of Faraday's basic law. We disagree. Let us see why, by considering the particular case of a uniform magnetic field:

It is also evident, by the results of the rotation of the wire and magnet (3097, 3106), that when a wire is moving amongst equal lines (or in a field of equal magnetic force), and with an uniform motion, then the current of electricity produced is proportionate to the *time*; and also to the *velocity* of motion.

They also prove, generally, that the quantity of electricity thrown into a current is directly as the amount of curves intersected.^{26, 27}

If a rigid conductor, perpendicular to the lines of magnetic force (and to the page), is moving in a uniform magnetic field, we can say that the electric field induced in the conductor is proportional to the component of its velocity along the direction perpendicular to the lines of magnetic force (Fig. 4):

²³ M. Faraday, *Experimental Researches in Electricity*, vol. I, London, 1839, p. 63, (218).

²⁴ This experiment has been performed for the following reasons: "Another point which I endeavoured to ascertain, was, wheter it was essential or not that the moving part of the wire should, in cutting the magnetic curves, pass into positions of greater or lesser magnetic force; or wether, always intersecting curves of equal magnetic intensity, the mere motion was sufficient for the production of the current. That the latter is true, has been proved already in several of the experiments on terrestrial magneto-electric induction. [...]" Ibidem, (217).

²⁵ According to L. P. Williams, in Faraday's diary, it is also reported that, by revolving only the magnet, there is no induced current. (L. P. Williams, *Michael Faraday*, Chapman & Hall, (1965), pp. 203-204. See also the statement "No mere rotation of a bar magnet on its axis, produces any induction effect on circuits exterior to it" contained in the paragraph 3090 of the *Researches* reported above. In my opinion, these Faraday's results has been overlooked. I have not found them in any contemporary or old textbook. At the XXXIX National Congress of the Italian Association of Physics Teachers (AIF) held in Milazzo in October 2000, Guido Pegna has presented, for teaching purposes, a nice experimental set up for reproducing all Faraday's experiments with the disc..

²⁶ M. Faraday, *Experimental Researches in Electricity*, vol. III, London, p. 346, (3114, 3115).

²⁷ If the current is proportional to the velocity of the conductor and the amount of electricity "thrown into a current" is proportional to the amount of curves intersected or, since the magnetic field is homogeneous, to the interval of time, the phrase "the current of electricity produced is proportionate to the *time*" is incongruent with the other two (if I do not misunderstand the whole statement).

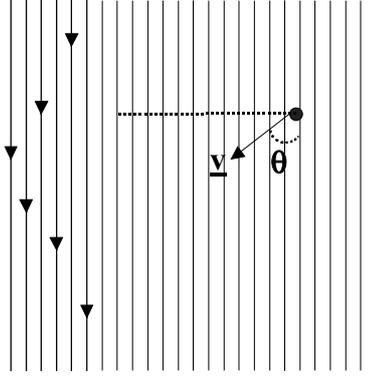


Fig. 4: a conductor in a magnetic field.

$$E_i \propto v \sin \vartheta$$

where ϑ is the angle between the velocity of the conductor and the lines of magnetic force. If we call B the proportionality factor, we get:

$$E_i \propto Bv \sin \vartheta$$

Finally, taking into account the relation between the direction of the velocity, of the lines of magnetic forces and of the induced current:

$$\vec{E}_i = \vec{v} \times \vec{B}$$

Which is *similar* to the expression of the magnetic component of Lorentz force: the only *relevant difference* being the fact that the velocity appearing in the above formula is the velocity of the conductor instead of the velocity of the charges contained in it. Hence, it appears that the physics of lines of magnetic force is, at least partially, expressed mathematically by the magnetic component of Lorentz force. Of course, a term of this type is not sufficient for describing all induction phenomena: we also need a term depending on the time variation of the magnetic field.

8.2 Maxwell

As it is well known, Maxwell repeatedly asserted that his *Treatise* was a mathematical translation of Faraday's physics. For instance:

[...] It was perhaps for the advantage of science that Faraday, though thoroughly conscious of the fundamental forms of space, time and force, was not a professed mathematician. He was not tempted to enter into the many interesting researches in pure mathematics which his discoveries would have suggested if they had been exhibited in a mathematical form, and he did not feel called upon either to force his results into a shape acceptable to the mathematical taste of the time, or to express them in a form which mathematicians might attack. He was thus left at leisure to do his proper work, to coordinate his ideas with his facts, and to express them in a natural, untechnical language.

It is mainly with the hope of making these ideas the basis of a mathematical method that I have undertaken this treatise.²⁸

However, as far as electromagnetic induction is concerned, Maxwell's treatment cannot be considered as a strict translation of Faraday's physics into mathematical form. In the introductory and descriptive part of his *Treatise* dedicated to the induction phenomena, Maxwell says:

The whole of these phenomena may be summed up in one law. When the number of lines of magnetic induction which pass through the secondary circuit in the positive direction is altered, an electromotive force acts round the circuit, which is measured by the rate of decrease of the magnetic induction through the circuit.²⁹

²⁸ J.C. Maxwell, *A treatise on electricity and magnetism*, third edition, vol. II, Dover Pub. Inc., 1954, p. 176, (528).

²⁹ J.C. Maxwell, *A treatise on electricity and magnetism*, third edition, vol. II, Dover Pub. Inc., 1954, p. 179, (531).

And:

Instead of speaking of the number of lines of magnetic force, we may speak of the magnetic induction through the circuit, or the surface-integral of magnetic induction extended over any surface bounded by the circuit.³⁰

In formula:

$$emf = -\frac{d\Phi(\vec{B})}{dt}$$

This is the “flux rule”: Maxwell does not write it down. The intersection of lines of magnetic force with the conductor (a true field description) is here substituted by the time variation of the magnetic flux through the circuit (a field description contaminated by an action at a distance; see section 4 above).

However, when Maxwell goes on to develop the “Theory of electric circuits”, he firstly defines the electrokinetic momentum p_2 of the induced (secondary) circuit as

$$p_2 = LI_2 + M_{21}I_1$$

where the subscripts 2 and 1 refer to the induced and the inducing circuit (primary) respectively; L and M_{21} are the self-inductance and the mutual inductance of the secondary and primary circuits; I_2 and I_1 their respective currents. *Primary and secondary circuits are at relative rest.*

Then, he states that the induced electromotive force is given by

$$emf = -\frac{dp_2}{dt}$$

and goes on to calculate p_2 . It turns out that

$$p_2 = \oint_l \vec{A} \cdot d\vec{l}$$

where \vec{A} is the vector potential. Therefore, the induced emf is given by:

$$emf = -\frac{d}{dt} \oint_l \vec{A} \cdot d\vec{l}$$

In the present case, in which the two circuits are at relative rest, this equation can be rigorously written as (using Stokes theorem)

$$emf = -\frac{d\Phi(\vec{B})}{dt}$$

i.e, the “flux rule”. Maxwell does not write it down.

The more interesting point comes out in the paragraph 598 entitled “*General equations of electromotive intensity*”. There, Maxwell allows the secondary (induced) circuit to move with respect the primary one and gets, for the equation of the electromotive intensity:

³⁰ Ibidem, pp. 188-189, (541).

$$\vec{E} = \vec{v} \times \vec{B} - \frac{\partial \vec{A}}{\partial t} - \text{grad} \varphi \quad (10)$$

and for the induced *emf*:

$$\text{emf} = \oint_i \left(\vec{v} \times \vec{B} - \frac{\partial \vec{A}}{\partial t} - \text{grad} \varphi \right) \cdot d\vec{l} \quad (11)$$

We have written Maxwell's equations using modern notation and in vector form: consequently, in the following quotation, we have made the necessary changes, since the original equations are for the vectors' components. Maxwell's comments:

The term involving the new quantity φ are introduced for sake of giving generality to the expression for \vec{E} . It disappears from the integral when extended round the closed circuit [...] The electromotive intensity has already been defined in Art. 68.³¹ It is also called the resultant electrical intensity, being the force which would be experienced by a unit of positive electricity placed at that point. *We have now obtained the most general value of this quantity in the case of a body moving in a magnetic field due to a variable electric system.*

If the body is a conductor, the electromotive force will produce a current; if it is a dielectric, the electromotive force will produce only electric displacement. The electromotive intensity, or the force on a particle, must be carefully distinguished from the electromotive force along an arc of a curve, the latter quantity being the line-integral of the former. See Art. 69.³²

And:

The electromotive intensity, given by equation (10), depends on three circumstances. The first of these is the motion of the particle through the magnetic field. The part of the force depending on this motion is expressed by the first term on the right of the equation. It depends on the velocity of the particle transverse to the lines of magnetic induction. [...] The second term in equation (10) depends on the time-variation of the magnetic field. This may be due either to the time-variation of the electric current in the primary circuit, or to motion of the primary circuit. [...] The last term is due to the variation of the function φ in different parts of the field.³³

Some comments.

- I) Maxwell says that the velocity which appear in equation (10) is the velocity of the *particle*. The definitions reported in note 31 suggest to understand the term *particle* as “a small body charged with the unit positive electricity”, though in the case of a material body the use of this term is not appropriate. Anyway, the calculation performed by Maxwell shows that the velocity we are speaking about is the velocity of an element of

³¹ We report here parts of the paragraphs 68 and 69 of volume I of the *Treatise*. “*Definition*. The resultant electric intensity at any point is the force which would be exerted on a small body charged with the unit positive electricity, if it were placed there without disturbing the actual distribution of electricity. This force tends to move a body charged with electricity, but to move the electricity within the body, so that the positive electricity tends to move in the direction of \vec{E} and the negative electricity in the opposite direction. Hence the quantity \vec{E} is also called the Electromotive Intensity at the point (x,y,z)” (68). “The electromotive force along a given arc AP of a curve is numerically measured by the work which would have been done the electric intensity on a unit of positive electricity carried along the curve from A, the beginning, to P, the end of the arc” (69).

³² Ref. 25, pp. 239-240, (598).

³³ Ibidem, pp.240-241, (599).

the induced (secondary) circuit. As a matter of fact, Maxwell did not have a model for the current, because he did not have a model for the electricity:

The electric current cannot be conceived except as a kinetic phenomenon. [...] The effects of the current, such as electrolysis, and the transfer of electrification from one body to another, are all progressive actions which require time for their accomplishment, and are therefore of the nature of motions.

As to the velocity of the current, we have shewn that we know nothing about it, it may be the tenth of an inch in an hour, or a hundred thousand miles in a second (Exp. Res., 1648). So far are we from knowing its absolute value in any case, that we do not even know whether what we call the positive direction is the actual direction of the motion or the reverse.³⁴

Now, we easily write that $\vec{J} = -ne\vec{v}$, where \vec{J} is the current density due to n electrons per unit volume if \vec{v} is their velocity. Maxwell could not write anything similar.

II) Apart from the meaning of \vec{v} , equation (11) is the same as equation (9) of our derivation. For Maxwell too, the “flux rule” is only a particular case of a more general law. However, Maxwell does not comment on this point.

9. We do not know

The “flashbacks” reported above suggest that the question “why the flux rule?” is, from a historical point of view, legitimate independently from the correctness of the general law presented in this paper.

In fact:

1. The Faraday’s physics of lines of magnetic force has not been completely translated into a mathematical form (of course, we are not sure that such a translation is possible).
2. The “flux rule” is not a mathematical translation of the physics of lines of magnetic force.
3. In Maxwell’s *Treatise*, the “flux rule” is accompanied by “a general equation of electromotive intensity” (10) with a correspondent “general equation of induced *emf*” (11). The velocity appearing in these equations is the velocity of the element of the circuit and not the velocity of the charges contained in it. However, equation (11) coincides with our equation (9) which is the general law of induction written for loops of wire (when the contribution of the charge drift velocity vanishes).
4. The explicit formulation of Lorentz force should have suggested a revision of all the subject.

I believe that there is a puzzling historical problem.

10. Physics and history

There is a basic difference between, say, social history and history of physics. Social history cannot change “the course of past events”. The physics of the past, instead, can be “changed” in the sense that it can be read in new ways and/or further developed, as a *living matter*, in such a way that new

³⁴ Ibidem, p. 210-211, (569). The quoted paragraph of Faraday’s *Experimental Researches* says: “As long as the terms *current* and *electro-dynamic* are used to express those relations of the electric forces in which progression of either fluids or effects are supposed to occur (283), so long will the idea of velocity be associated with them; and this will, perhaps, be more especially the case if the hypothesis of a fluid or fluids be adopted”. And, in Faraday’s paragraph 283: “By *current*, I mean anything progressive, whether it be a fluid of electricity, or two fluids moving in opposite directions, or merely vibrations, or, speaking more generally, progressive forces. [...]”.

knowledge is produced. Moreover, a reliable evaluation of Science's achievements requires a historical perspective. For these reasons, physicists should know about the basic historical features of their discipline.

There are many ways of doing history of physics; we are speaking of those ways that deal, mainly if not exclusively, with the developments of theories and experiments. Historians of this kind should know very well the physics they are speaking of; if they do not, their work becomes much harder and they miss the chance of rendering the physics of the past live again. Moreover, when the physics is cloudy the historian, like the physicist, may be disguised.

To paraphrase again a well known sentence: Physics without History is blind (we must know where we come from, if we want to understand who are we and where are we going); History of Physics without (good) Physics is void.